Kunming School 2018: Lab assignments day 4

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1 Write your own 1D Euler Riemann problem solver

The 'exact' solution to the Riemann problem for 1D hydro can be numerically computed as follows (see also Toro, 1999, Riemann Solvers and Numerical Methods for Fluid dynamics, chapter 4). We seek the solution at all times t of the initial condition at $t = 0$ where a left state $\mathbf{U}_l = (v_l, p_l, \rho_l)$ is seperated from a right state $\mathbf{U}_r = (v_r, p_r, \rho_r)$, where the states $\mathbf{U}$ contain velocity, pressure and density, respectively. The pressure in the intermediate region p_* can be found as the zero of the function

$$f(p, \mathbf{U}_l, \mathbf{U}_r) = f_l(p, \mathbf{U}_l) + f_r(p, \mathbf{U}_r) + v_r - v_l.$$

In this expression, we have

$$f_k(p, \mathbf{U}_k) = \begin{cases} (p - p_k) \left[\frac{\frac{2}{(\gamma+1)\rho_k}}{p + \frac{(\gamma-1)p_k}{\gamma+1}} \right]^{\frac{1}{2}} & \text{if} \quad p > p_k \ \ (shock) \\ \frac{2c_k}{\gamma-1} \left[\left(\frac{p}{p_k} \right)^{\frac{\gamma-1}{2\gamma}} - 1 \right] & \text{if} \quad p \leq p_k \ \ (rarefaction) \end{cases}$$

The velocity in the middle region is then

$$v_* = \frac{v_l + v_r}{2} + \frac{f_r(p_*) - f_l(p_*)}{2}.$$

This zero can be computed using a Newton-Raphson scheme on the function $f(p)$, with start value e.g. $p_0 = 0.5(p_r + p_l)$ (better start values may exist, and may need to be used to avoid the generation of negative pressure values!). Another way to find the zero is to note that with $p_{\min} = \min(p_l, p_r)$ and $p_{\max} = \max(p_l, p_r)$ the signs of $f(p_{\min})$ and $f(p_{\max})$ will determine which bracket to use for the zero p_*: it lies in $[0, p_{\min}]$ when $f(p_{\min}) > 0$ and $f(p_{\max}) > 0$; it is $\in [p_{\min}, p_{\max}]$ when their signs differ; and p_* is in $[p_{\max}, +\infty]$ when $f(p_{\min}) < 0$ and $f(p_{\max}) < 0$. These statements can be made since $f(p)$ can be shown to be a monotonous function of p. There is a caveat: pressure positivity requires $f(0) < 0$ leading to (using sound speeds where $c_k^2 = \gamma p_k / \rho_k$)

$$\frac{2c_l}{\gamma - 1} + \frac{2c_r}{\gamma - 1} > v_r - v_l\,.$$

If this condition is not fulfilled, vacuum will be created (and this case needs special treatment). Once the initial bracket for the zero is identified, a simple bisection algorithm can be employed as well to zoom in on the root of $f(p)$. As always, a root for a nonlinear function is computed numerically to within a certain accuracy, hence the quotation marks for denoting this as an 'exact' solution. As seen from the above formula, whether we have a left shock or rarefaction ultimately depends on $p_* > p_l$ or $p_* < p_l$, respectively. Similarly, the character of the right nonlinear wave is determined by which pressure is dominant: p_* or p_r.

With knowledge of $\mathbf{U}_l$, $\mathbf{U}_r$, p_* and v_* as well as the nature of the left and right nonlinear wave, we can then fully compute the solution as function of x at any time t (it will be a self-similar solution, essentially depending on x/t if the initial discontinuity is placed at $x = 0$). For a shock, we need to determine its shock speed (s_1 or s_3), and the density across the shock. These follow both from Rankine Hugoniot, and in essence use the following equations:

$$\frac{v_l - s}{v_r - s} = \frac{\rho_r}{\rho_l}, \tag{1}$$

$$(s - v_l)^2 = c_l^2 \left[1 + \frac{\gamma + 1}{2\gamma} \left(\frac{p_r}{p_l} - 1 \right) \right]. \tag{2}$$

If we know the signal is a rarefaction, we can use the knowledge of the generalized Riemann invariants across these continuous waves to deduce the density (since entropy or $S = p\rho^{-\gamma}$ is constant) in the middle (left or right) state. The rarefaction wave itself will have a tail and head position which are given by $v_k \pm c_k$ and $v_* \pm c_*$ where $c_* = \sqrt{\gamma p_*/\rho_*}$. The variation through the rarefaction is then also found in function of x/t from the knowledge of the Riemann invariants on the rays $x/t = v \pm c$. In particular, for a left rarefaction one then finds:

$$\rho = \rho_l \left[\frac{2}{\gamma + 1} + \frac{\gamma - 1}{(\gamma + 1) c_l} \left(v_l - \frac{x}{t} \right) \right]^{\frac{2}{\gamma - 1}}, \tag{3}$$

$$v = \frac{2}{\gamma + 1} \left[c_l + \frac{\gamma - 1}{2} v_l + \frac{x}{t} \right], \tag{4}$$

$$p = p_l \left[\frac{2}{\gamma + 1} + \frac{\gamma - 1}{(\gamma + 1) c_l} \left(v_l - \frac{x}{t} \right) \right]^{\frac{2\gamma}{\gamma - 1}}. \tag{5}$$

Deduce these expressions, find the corresponding expressions for the variation through a right rarefaction, and consecutively write a little plotting program which shows the variation of ρ, v, p and the 3 Riemann Invariants

$$\mathcal{R} = (v - \frac{2c}{\gamma - 1}, S = p\rho^{-\gamma}, v + \frac{2c}{\gamma - 1})$$

at any point x in a (finite) domain $[x_{\min}, x_{\max}]$ at any time t, for a Riemann problem with given left and right state $\mathbf{U}_l$ and $\mathbf{U}_r$ seperated at the middle of the interval for $t = 0$. An extra input parameter is the ratio of specific heats γ, and the number of points used in between $[x_{\min}, x_{\max}]$ to produce the plots.

2 Use MPI-AMRVAC for 2D (or 3D) hydro setups

Explore the effect of using different limiters, different temporal discretizations, and/or different shock-capturing schemes (TVDLF, HLL, HLLC) on a specific

2D hydro setup. Do this at first on uniform grids, and explore the tendency with increasing resolutions. Finally, activate AMR and see if solution details are captured correctly (but are computed faster).